

CONCURRENCE FOR INFINITE-DIMENSIONAL QUANTUM SYSTEMS

YU GUO, JINCHUAN HOU, AND YUNCAI WANG

ABSTRACT. Concurrence is an important entanglement measure for states in finite-dimensional quantum systems that was explored intensively in the last decade. In this paper, we extend the concept of concurrence to infinite-dimensional bipartite systems and show that it is continuous and does not increase under local operation and classical communication (LOCC). Moreover, based on the partial Hermitian conjugate (PHC) criterion proposed in [Chin. Phys. Lett. **26**, 060305(2009); Chin. Sci. Bull. **56**(9), 840–846(2011)], we introduce a concept of the PHC measure and show that it coincides with the concurrence, which provides another perspective on the concurrence.

1. INTRODUCTION

Entanglement, being viewed as one of the key features of quantum world that has no classical counterpart, is perhaps *the most challenging subject of modern quantum theory*. There are two distinct directions for characterizing entanglement. One is to find proper criteria of detecting entanglement, and the other is to find a “good” entanglement measure, namely, to define the best measure quantifying an amount of entanglement of a given state. Among a number of entanglement measures, *concurrence* is a subject of intense research interest [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18], which has been shown to play a key role in analyzing the ultrabright source of entangled photon pairs [19], describing quantum phase transitions in various interacting quantum many-body systems [20], affecting macroscopic properties of solids significantly [21], exploring dynamics of entanglement for noisy qubits that make dipole-dipole interaction [22] and revealing distinct scaling behavior for different types of multipartite entanglement [23], etc.

Concurrence is originally derived from the entanglement of formation (EOF) which is used to compute the amount of entanglement for pure states in two-qubit systems [1]. Because of the EOF is a monotonically increasing function of the concurrence, thus the concurrence itself can also be regarded as an entanglement measure. Afterward, the concept of concurrence was extended to two-qubit mixed states by means of convex roof construction [2], and then, to arbitrary but finite-dimensional bipartite as well as multipartite systems for both pure and mixed states [5, 7].

The continuous-variable systems can also be used for quantum information processing and quantum computing [24]. Most analysis of entanglement in continuous-variable systems relies on expressing the states of the system in terms of some discrete but infinite basis. Then, the

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following problems arisen naturally: Can the concept of concurrence be extended to infinite-dimensional case? Is it also a “well-defined” entanglement measure? In the present paper, we answer these questions affirmatively.

In [25], the *partial Hermitian conjugate* (PHC) criterion for pure states in finite-dimensional systems was proposed and then generalized in [26] to infinite-dimensional case. The (PHC) criterion says that: A bipartite pure state is separable if and only if it is PHC invariant (see below). The authors of [26, 25] also pointed out that one may obtain an entanglement measure from (PHC) criterion since, for any entangled pure state, the PHC of it is not equal to itself, and thus the trace norm or the Hilbert-Schmidt norm of the difference between them may be an entanglement measure of the given state. Interestingly, as what we will show, underlying the Hilbert-Schmidt norm, the induced entanglement measure, *PHC measure*, coincides with concurrence. This result makes contribution to the more profound understanding of the concurrence.

In this paper, we consider the bipartite system consisting of two parties A and B which are associated with the state spaces H_A and H_B , respectively, with $\dim H_A \otimes H_B \leq +\infty$. We denote by ρ_A and ρ_B the reduced density operators of ρ with respect to the subsystems A and B, respectively, i.e., $\rho_A = \text{Tr}_B(\rho)$ and $\rho_B = \text{Tr}_A(\rho)$. A bipartite state ρ acting on $H = H_A \otimes H_B$ is called *separable* if it can be written as

$$(1) \quad \rho = \sum_i p_i \rho_i^A \otimes \rho_i^B, \quad \sum_i p_i = 1, \quad p_i \geq 0$$

or it is a limit of the states of the above form under the trace norm topology [33], where ρ_i^A and ρ_i^B are pure states on the subsystems associated to the Hilbert spaces H_A and H_B , respectively. A state that is not separable is said to be *entangled*. Particularly, if a state can be represented in the form as in Eq.(1), it is called *countably separable* [34]. It is worth mentioning that, with increasing state space dimension, quantifying entanglement becomes more and more difficult to implement in practice.

The structure of this paper is as follows. In Sec.II we extend the concept of the concurrence to infinite-dimensional bipartite systems and show that it is a continuous function under the trace-class norm topology. This result is new even for finite-dimensional case, and enables us to prove that the concurrence is also a well-defined monotonic entanglement measure for infinite-dimensional case. Going further, another entanglement measure which is closely related to concurrence, *tangle*, is investigated. The PHC measure is introduced and discussed in Sec.III. A brief conclusion is given in the last section.

2. CONCURRENCE FOR INFINITE-DIMENSIONAL BIPARTITE STATES

We start by reviewing some results from finite-dimensional cases. For the bipartite pure state $|\psi\rangle \in H_A \otimes H_B$ with $\dim H_A \otimes H_B < +\infty$, the concurrence $C(|\psi\rangle)$ of $|\psi\rangle$ is defined in [7] by

$$(2) \quad C(|\psi\rangle) = \sqrt{2[1 - \text{Tr}(\rho_A^2)]},$$

where $\rho_A = \text{Tr}_B(|\psi\rangle\langle\psi|)$. Equivalently,

$$C(|\psi\rangle) = \sqrt{\sum_{i,j,k,l} |a_{ik}a_{jl} - a_{il}a_{jk}|^2}$$

provided that $|\psi\rangle = \sum_{i,j} a_{ij}|i\rangle|j'\rangle$, where $\{|i\rangle\}$ and $\{|j'\rangle\}$ are given orthonormal bases of H_A and H_B , respectively. The concurrence is extended to mixed states by means of convex roof construction [27],

$$(3) \quad C(\rho) = \min_{\{p_i, |\psi_i\rangle\}} \left\{ \sum_i p_i C(|\psi_i\rangle) \right\},$$

where the minimum is taken over all possible ensembles of ρ (here, $\{p_i, |\psi_i\rangle\}$ is called an ensemble of ρ whenever $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ with $\{p_i\}$ a probability distribution and $\{|\psi_i\rangle\}$ a family of pure states).

The tangle is another measure closely related to the concurrence. The tangle $\tau(|\psi\rangle)$ for pure state $|\psi\rangle$ is defined by $\tau(|\psi\rangle) = C^2(|\psi\rangle)$, and the tangle for mixed state ρ is defined by

$$(4) \quad \tau(\rho) = \min_{\{p_i, |\psi_i\rangle\}} \left\{ \sum_i p_i C^2(|\psi_i\rangle) \right\}$$

(Ref. [28]). Note that, although the tangle and the concurrence are equivalent to each other as entanglement measures for pure states, they are different for mixed states. In fact, it holds that $\tau(\rho) \geq C^2(\rho)$ and the equality holds in the case of two-qubit states [29]. It is evident that ρ is separable if and only if $C(\rho) = \tau(\rho) = 0$.

With the same spirit in mind, we extend the concepts of concurrence and tangle to infinite-dimensional bipartite systems.

Definition 1. Let $|\psi\rangle \in H_A \otimes H_B$ with $\dim H_A \otimes H_B = +\infty$ be a pure state.

$$(5) \quad C(|\psi\rangle) := \sqrt{2(1 - \text{Tr}(\rho_A^2))},$$

is called the concurrence of $|\psi\rangle$, where $\rho_A = \text{Tr}_B(|\psi\rangle\langle\psi|)$.

Since the eigenvalues of ρ_A coincide with that of $\rho_B = \text{Tr}_A(|\psi\rangle\langle\psi|)$, with no loss of generality, we always use the reduced density operators with respect to the subsystem A. It is clear that $C(|\psi\rangle) = 0$ if and only if $|\psi\rangle$ is separable.

For a mixed state ρ , the concurrence of ρ can be defined by means of the generalized convex roof construction, namely,

$$(6) \quad C(\rho) := \inf_{\{p_i, |\psi_i\rangle\}} \left\{ \sum_i p_i C(|\psi_i\rangle) \right\},$$

where the infimum is taken over all possible ensembles $\{p_i, |\psi_i\rangle\}$ of ρ .

The following proposition provides two computational formulas of the concurrence for pure states.

Proposition 1 Let $|\psi\rangle \in H_A \otimes H_B$ with $\dim H_A \otimes H_B = +\infty$ be a pure state.

(1) If the Fourier expansion of $|\psi\rangle$ with respect to some given product basis $\{|i\rangle|j'\rangle\}$ of $H_A \otimes H_B$ is $|\psi\rangle = \sum_{i,j} a_{ij}|i\rangle|j'\rangle$, then

$$(7) \quad C(|\psi\rangle) = \sqrt{\sum_{i,j,k,l} |a_{ik}a_{jl} - a_{il}a_{jk}|^2}.$$

(2) If the Schmidt decomposition of $|\psi\rangle$ is $|\psi\rangle = \sum_k \lambda_k |k\rangle|k'\rangle$, then

$$(8) \quad C(|\psi\rangle) = \sqrt{2 \sum_{k \neq l} \lambda_k^2 \lambda_l^2}.$$

Proof (1) Consider the operator $D = D_{|\psi\rangle} = (a_{ij}) : H_B \rightarrow H_A$ defined by $D|j'\rangle = \sum_i a_{ij}|i\rangle$. Since $\text{Tr}(DD^\dagger) = \sum_{i,j} |a_{ij}|^2 = 1$, D is a Hilbert-Schmidt operator. With $\rho = |\psi\rangle\langle\psi|$, it is easily checked that $\rho_A = DD^\dagger$. As $\text{Tr}((DD^\dagger)^2) = \sum_{i,j,k,l} a_{ik}\bar{a}_{il}a_{jl}\bar{a}_{jk}$, we have

$$\begin{aligned} 1 - \text{Tr}(\rho_A^2) &= \left(\sum_{i,j} a_{ij}\bar{a}_{ij}\right)^2 - \sum_{i,j,k,l} a_{ik}\bar{a}_{il}a_{jl}\bar{a}_{jk} \\ &= \sum_{i,j,k,l} (a_{ik}\bar{a}_{ik}a_{jl}\bar{a}_{jl} - a_{ik}\bar{a}_{il}a_{jl}\bar{a}_{jk}) \\ &= \frac{1}{2} \sum_{i,j,k,l} (a_{ik}a_{jl} - a_{il}a_{jk})(\bar{a}_{ik}\bar{a}_{jl} - \bar{a}_{il}\bar{a}_{jk}) \\ &= \frac{1}{2} \sum_{i,j,k,l} |a_{ik}a_{jl} - a_{il}a_{jk}|^2. \end{aligned}$$

Hence $C(|\psi\rangle) = \sqrt{2(1 - \text{Tr}(\rho_A^2))} = \sqrt{\sum_{i,j,k,l} |a_{ik}a_{jl} - a_{il}a_{jk}|^2}$, as desired.

(2) can be checked similarly and we omit its proof here. \square

It is known that the concurrence is an entanglement measure for finite dimensional systems since it meets the following conditions:

- (i) $E(\rho) = 0$ if and only if ρ is separable;
- (ii) $E(\rho) = E(U_A \otimes U_B \rho U_A^\dagger \otimes U_B^\dagger)$ holds for any local unitary operators U_A and U_B on the subsystems H_A and H_B , respectively;
- (iii) E is LOCC monotonic, i.e., $E(\Lambda(\rho)) \leq E(\rho)$ holds for any local operation and classical communication (LOCC) Λ [6].

The conditions (i)-(iii) above are necessary for any entanglement measure E [30]. Generally, an entanglement measure E also satisfies

- (iv) $E(\sum_i p_i \rho_i) \leq \sum_i p_i E(\rho_i)$ for mixed state $\rho = \sum_i p_i \rho_i$, where $p_i \geq 0$, $\sum_i p_i = 1$ (see in [31]).

If (iii)-(iv) are satisfied by an entanglement measure E , then it is called an entanglement monotone [32].

In what follows we show that the concurrence C defined in Eq.(4)-(5) for infinite-dimensional systems is also an entanglement monotone, i.e., (i)-(iv) are satisfied by C for infinite-dimensional case as well.

Checking C meets Condition (ii) is straightforward.

The condition (i) is obviously satisfied by the concurrence for finite-dimensional case. This is because that, every separable state ρ in a finite-dimensional bipartite system is countably separable, that is, there exists an ensemble $\{p_i, |\psi_i\rangle\}$ of ρ such that $|\psi_i\rangle$ s are separable pure states and thus we get immediately that $0 \leq C(\rho) \leq \sum_i p_i C(|\psi_i\rangle) = 0$ as $C(|\psi_i\rangle) = 0$. However, the fact that ρ is separable implies $C(\rho) = 0$ is not obvious anymore for infinite-dimensional case since there do exist some separable states in infinite-dimensional systems that are not countably separable [34]. For such separable states that are not *countably separable*, there doesn't exist any ensemble $\{p_i, |\psi_i\rangle\}$ of ρ such that $|\psi_i\rangle$ s are separable and one can not get $C(\rho) = 0$ directly. It is clear that, if C is continuous, then $C(\rho) = 0$ whenever ρ is separable because it is a limit of countably separable states.

The continuity of the concurrence C is established in Proposition 2, which is not obvious even for finite-dimensional systems.

Proposition 2 The concurrence is continuous for both finite- and infinite-dimensional systems, i.e.,

$$(9) \quad \lim_{n \rightarrow \infty} C(\rho_n) = C(\rho) \quad \text{whenever} \quad \lim_{n \rightarrow \infty} \rho_n = \rho$$

in the trace-norm topology.

Proof To prove the continuity of C , let us extend the concurrence of states to that of self-adjoint trace-class operators.

Let A be a self-adjoint trace-class operators acting on $H_A \otimes H_B$. We define the concurrence of A by

$$C(A) = \text{Tr}(|A|) C\left(\frac{|A|}{\text{Tr}(|A|)}\right),$$

where $|A| = (A^\dagger A)^{\frac{1}{2}}$. It is clear that

$$C(A) = \inf_{\{\lambda_i, |\psi_i\rangle\}} \sum_i \lambda_i C(|\psi_i\rangle),$$

where the infimum is taken over all $\{\lambda_i, |\psi_i\rangle\}$ with $\lambda_i \geq 0$, $\sum_i \lambda_i = \text{Tr}(|A|)$ and $|A| = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$. It is an immediate consequence of the definition that, if $0 \leq |A| \leq |B|$, then $C(A) \leq C(B)$.

Assume that $\rho_n, \rho \in \mathcal{S}(H_A \otimes H_B)$ and $\lim_{n \rightarrow \infty} \rho_n = \rho$. Let $\vartheta_n = \rho - \rho_n$ and let

$$\vartheta_n = \sum_{k(n)} \lambda_{k(n)} |\eta_{k(n)}\rangle\langle\eta_{k(n)}|$$

be its spectral decomposition.

We claim that

$$(10) \quad C(\rho) = C(\rho_n + \vartheta_n) \leq C(\rho_n) + C(\vartheta_n).$$

For any $\varepsilon > 0$, there exist ensembles $\{p_{k(n)}, |\psi_{k(n)}\rangle\}$ and $\{q_{l(n)}, |\phi_{l(n)}\rangle\}$ of ρ_n and $|\vartheta_n|$, respectively, and $0 < \epsilon_1, \epsilon_2 < \varepsilon$, such that

$$C(\rho_n) = \sum_{k(n)} p_{k(n)} C(|\psi_{k(n)}\rangle) - \frac{\epsilon_1}{2}$$

and

$$C(|\vartheta_n|) = \sum_{l(n)} q_{l(n)} C(|\phi_{l(n)}\rangle) - \frac{\epsilon_2}{2}.$$

We compute

$$\begin{aligned} C(\rho_n + \vartheta_n) &\leq C(\rho_n + |\vartheta_n|) \\ &\leq \sum_{k(n)} p_{k(n)} C(|\psi_{k(n)}\rangle) + \sum_{l(n)} q_{l(n)} C(|\phi_{l(n)}\rangle) \\ &= C(\rho_n) + C(\vartheta_n) + \frac{\epsilon_1 + \epsilon_2}{2}. \end{aligned}$$

Since ε is arbitrarily given, the claim is proved.

Similarly, using $C(\rho_n) = C(\rho - \vartheta_n) \leq C(\rho + |\vartheta_n|)$, we obtain

$$C(\rho_n) \leq C(\rho) + C(|\vartheta_n|),$$

which, together with Eq.(10), implies that

$$|C(\rho_n) - C(\rho)| \leq C(|\vartheta_n|).$$

Observing that $C(\vartheta_n) \rightarrow 0$ ($n \rightarrow \infty$) since $C(\vartheta_n) \leq \sum_{k(n)} \sqrt{2} |\lambda_{k(n)}|$ and $\text{Tr}(|\vartheta_n|) = \sum_{k(n)} |\lambda_{k(n)}| \rightarrow 0$, we get $\lim_{n \rightarrow \infty} C(\rho_n) = C(\rho)$, as desired. \square

We now begin to check that C satisfies properties (iii)-(iv). For finite-dimensional case, Vidal [32] proposed a nice recipe for determining entanglement monotones by proving that the convex roof extension of a pure state measure E satisfying the two conditions below is an entanglement monotone (Ref. [32, Theorem 2]):

(a) For a pure state $|\psi\rangle$, $\rho_A = \text{Tr}_B(|\psi\rangle\langle\psi|)$, define a function f by $f(\rho_A) = E(|\psi\rangle)$, then

$$f(U\rho_A U^\dagger) = f(\rho_A);$$

and

(b) f is concave, namely,

$$f(\lambda\rho_1 + (1-\lambda)\rho_2) \geq \lambda f(\rho_1) + (1-\lambda)f(\rho_2)$$

for any density matrices ρ_1, ρ_2 , and any $0 \leq \lambda \leq 1$.

For infinite-dimensional bipartite systems, every LOCC admits a form of

$$(11) \quad \Lambda(\rho) = \sum_{i=1}^N (A_i \otimes B_i) \rho (A_i^\dagger \otimes B_i^\dagger)$$

with $\sum_{i=1}^N A_i^\dagger A_i \otimes B_i^\dagger B_i \leq I_A \otimes I_B$, where N may be $+\infty$ and the series converges in the strong operator topology [35]. Let $\mathcal{S}(H_A \otimes H_B)$ be the set of all quantum states acting on $H_A \otimes H_B$. According to the entanglement monotone scenario discussed in [32], in order to prove that

a function $E : \mathcal{S}(H_A \otimes H_B) \rightarrow \mathbb{R}^+$ satisfying (i)-(ii) is LOCC monotonic, we only need to consider the sequence of LOCC $\{\Lambda_{B,k}\}$ or $\{\Lambda_{A,l}\}$, of the form

$$(12) \quad \Lambda_{B,k}(\rho) = \sum_{i(k)} (I_A \otimes B_{i(k)}) \rho (I_A \otimes B_{i(k)}^\dagger)$$

or

$$(13) \quad \Lambda_{A,l}(\rho) = \sum_{j(l)} (A_{j(l)} \otimes I_B) \rho (A_{j(l)}^\dagger \otimes I_B),$$

where $\sum_{i(k)} B_{i(k)}^\dagger B_{i(k)} \leq I_B$ and $\sum_{j(l)} A_{j(l)}^\dagger A_{j(l)} \leq I_A$ (here, the series converges in the strong operator topology) with $\sum_k \text{Tr}(\Lambda_{B,k}(\rho)) = \sum_l \text{Tr}(\Lambda_{A,l}(\rho)) = 1$, $B_{i(k)}$ s (resp. $A_{j(l)}$ s) are operators from H_B (resp. H_A) into $H_{B'}$ (resp. $H_{A'}$) for some Hilbert space $H_{B'}$ (resp. $H_{A'}$), and where k (resp. l) labels different outcomes if at some stage of local manipulations part B (resp. A) performs a measurement. With no loss of generality, hereafter we consider the LOCC $\{\Lambda_{B,k}\}$ as in Eq.(12). Applying $\Lambda_{B,k}$ to ρ , the state becomes

$$\rho'_k = \frac{\Lambda_{B,k}(\rho)}{p_k}$$

with probability $p_k = \text{Tr}(\Lambda_{B,k}(\rho))$. Therefore, the final state is $\rho' = \sum_k p_k \rho'_k$. By [32], if $E(\rho') \leq E(\rho)$ holds for $\Lambda_{B,k}$, then the condition (iii) holds for E . We show below that, for infinite-dimensional case, if E is continuous on quantum states under the trace norm topology, then (a)-(b) are sufficient conditions for E to be an entanglement monotone as well.

Proposition 3 Let E be an entanglement measure for pure states in infinite-dimensional systems and define $E(\rho) := \inf_{\{p_i, |\psi_i\rangle\}} \{\sum_i p_i E(|\psi_i\rangle)\}$ for mixed state ρ . Let $f(\rho_A) = E(|\psi\rangle\langle\psi|)$, $\rho_A = \text{Tr}_B(|\psi\rangle\langle\psi|)$. Assume that E is continuous and f satisfying (a)-(b). Then E is an entanglement monotone, i.e., E satisfying (iii)-(iv).

Proof We assume that f satisfy conditions (a) and (b), namely (a) $f(U\rho_A U^\dagger) = f(\rho_A)$ for any unitary operators on H_A and (b) f is concave.

By (a), we know that $E(\rho)$ is invariant under local unitary operations, i.e., $E(U_A \otimes U_B \rho U_A^\dagger \otimes U_B^\dagger) = E(\rho)$ for any unitary operators U_A and U_B acting on H_A and H_B respectively (notice that condition (iii) implies that $E(\rho)$ is invariant under local unitary operations).

In what follows, we show that (iii) holds for E and LOCC $\{\Lambda_{B,k}\}$, from which, according to the entanglement monotone scenario proposed in [32], we can thus obtain that (iii) holds for E and any LOCC Λ .

We assume first that ρ is a pure state, $\rho = |\psi\rangle\langle\psi|$. If part B performs $\Lambda_{B,k}$ on subsystem B as in Eq.(11), then the state becomes $\rho'_k = \frac{\Lambda_{B,k}(\rho)}{p_k}$ with probability $p_k = \text{Tr}(\Lambda_{B,k}(\rho))$. Writing $\rho'_{A,k} = \text{Tr}_B(\rho_k)$, we obtain $\rho_A = \sum_k p_k \rho'_{A,k}$. For any ensemble $\{r_{kl}, |\psi_{kl}\rangle\}$ of ρ'_k , we have

$$E(\rho'_k) \leq \sum_l r_{kl} E(|\psi_{kl}\rangle).$$

It yields

$$\begin{aligned}
E(\rho) &= f(\rho_A) = f\left(\sum_k p_k \rho'_{A,k}\right) \\
&= f\left(\sum_{k,l} p_k r_{kl} \rho'_{A,kl}\right) \geq \sum_{k,l} p_k r_{kl} f(\rho'_{A,kl}) \\
&= \sum_{k,l} p_k r_{kl} E(|\psi_{kl}\rangle) \geq \sum_k p_k E(\rho'_k),
\end{aligned}$$

where $\rho'_{A,kl} = \text{Tr}_B(|\psi_{kl}\rangle\langle\psi_{kl}|)$, the first inequality holds since f is concave and continuous. Therefore, (iii) is satisfied by E if ρ is pure.

Assume that ρ is mixed. Performing $\Lambda_{B,k}$ on ρ and denote $\rho'_k = \frac{\Lambda_{B,k}(\rho)}{p_k}$ with probability $p_k = \text{Tr}(\Lambda_{B,k}(\rho))$. Observe that, for any $\varepsilon > 0$, there exists an ensemble $\{t_j, |\eta_j\rangle\}$ of ρ , and $0 < \epsilon_1 < \varepsilon$ such that

$$E(\rho) = \sum_j t_j E(|\eta_j\rangle) - \frac{\epsilon_1}{2}.$$

For each j , let

$$\rho'_{jk} = \frac{1}{t_{jk}} \Lambda_{B,k}(|\eta_j\rangle\langle\eta_j|),$$

where $t_{jk} = \text{Tr}(\Lambda_{B,k}(|\eta_j\rangle\langle\eta_j|))$. Then

$$\rho'_k = \frac{1}{p_k} \sum_j t_j t_{jk} \rho'_{jk}$$

and

$$E(|\eta_j\rangle) \geq \sum_k t_{jk} E(\rho'_{jk})$$

by what proved for pure states above. For each pair (j, k) , suppose that $\{t_{jkl}, |\psi_{jkl}\rangle\}$ is an ensemble of ρ'_{jk} such that

$$E(\rho'_{jk}) = \sum_l t_{jkl} E(|\psi_{jkl}\rangle) - \frac{\epsilon_{jk}}{2}, \quad 0 < \epsilon_{jk} < \frac{\varepsilon}{2^k}.$$

We achieve that

$$\begin{aligned}
E(\rho) &= \sum_j t_j E(|\eta_j\rangle) - \frac{\epsilon_1}{2} \\
&\geq \sum_{j,k} t_j t_{jk} E(\rho'_{jk}) - \frac{\epsilon_1}{2} \\
&= \sum_{j,k,l} t_j t_{jk} t_{jkl} E(|\psi_{jkl}\rangle) - \epsilon' \\
&\geq \sum_k p_k E(\rho'_k) - \epsilon'
\end{aligned}$$

for some $\epsilon' < \varepsilon$. Since ε is arbitrarily given, we see that (iii) is satisfied for mixed states as well.

Now we show that (iv) is valid. Let $\rho = \sum_k p_k \rho_k$. For any given $\varepsilon > 0$, there exists an ensemble of ρ_k , $\{q_{kl}, |\phi_{kl}\rangle\}$, and $0 < \epsilon < \varepsilon$ such that

$$E(\rho_k) \geq \sum_l q_{kl} E(|\phi_{kl}\rangle) - \frac{\epsilon}{2^k}.$$

As $\{p_k q_{kl}, |\phi_{kl}\rangle\}_{k,l}$ is an ensemble of ρ , this entails that

$$E(\rho) \leq \sum_k p_k \sum_l q_{kl} E(|\phi_{kl}\rangle) \leq \sum_k p_k E(\rho_k) + \varepsilon,$$

from which we see that $E(\rho) \leq \sum_k p_k E(\rho_k)$, finishing the proof. \square

Based on Proposition 3, we show below that the concurrence for infinite-dimensional systems defined in Eqs.(4)-(5) satisfies conditions (i)-(iv) and thus it is a well-defined entanglement measure (monotone).

Theorem 1 The concurrence defined in Eqs.(4)-(5) is an entanglement monotone.

Proof By Proposition 2 and Proposition 3, we only need to verify that the function f defined by $f(\rho_A) = C(|\psi\rangle)$ with $\rho_A = \text{Tr}_B(|\psi\rangle\langle\psi|)$ satisfies (a) and (b). Note that, for any $\rho \in \mathcal{S}(H_A)$, we have $f(\rho) = \sqrt{2(1 - \text{Tr}(\rho^2))}$. Thus (a) is obvious. We check that f is concave. For any given states ρ_1 and ρ_2 on H_A , let

$$\rho = \lambda \rho_1 + (1 - \lambda) \rho_2, \quad 0 \leq \lambda \leq 1.$$

Then

$$f(\lambda \rho_1 + (1 - \lambda) \rho_2) \geq \lambda f(\rho_1) + (1 - \lambda) f(\rho_2)$$

if and only if

$$\text{Tr}(\rho_1^2) + \text{Tr}(\rho_2^2) \geq 2\text{Tr}(\rho_1 \rho_2).$$

But the last inequality is always valid. Thus, f is concave. \square

With the same spirit as that for finite-dimensional case, we define the tangle of a pure state in the case of infinite-dimensional systems by

$$(14) \quad \tau(|\psi\rangle) = C^2(|\psi\rangle).$$

If $|\psi\rangle = \sum_k \lambda_k |k\rangle |k'\rangle$ is the Schmidt decomposition of $|\psi\rangle$ [26], then

$$(15) \quad \tau(|\psi\rangle) = 2(1 - \text{Tr}(\rho_A^2)) = 2 \sum_{k \neq l} \lambda_k^2 \lambda_l^2.$$

The tangle of a mixed state ρ , $\tau(\rho)$ can be naturally defined by

$$(16) \quad \tau(\rho) := \inf_{\{p_i, |\psi_i\rangle\}} \left\{ \sum_i p_i C^2(|\psi_i\rangle) \right\},$$

where the infimum is taken over all possible ensembles $\{p_i, |\psi_i\rangle\}$ of ρ . By Proposition 2 and Theorem 1, τ is continuous and satisfies the conditions (i)-(iv) as well. Therefore, τ is a good entanglement measure, too.

For mixed state ρ , $C(\rho) \neq \sqrt{2[1 - \text{Tr}(\rho_A^2)]}$ in general. For the finite-dimensional case, it is showed in [9, 36] that $C^2(\rho) \leq 2[1 - \text{Tr}(\rho_A^2)]$. In fact, we have the following result.

Proposition 4 Let $\rho \in \mathcal{S}(H_A \otimes H_B)$ with $\dim H_A \otimes H_B \leq \infty$. Then

$$(17) \quad C^2(\rho) \leq \tau(\rho) \leq 2[1 - \text{Tr}(\rho_A^2)].$$

Proof For any $\epsilon > 0$, there exists $\{p_i, |\psi_i\rangle\}$ such that $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ and

$$\tau(\rho) \geq \sum_i p_i C^2(|\psi_i\rangle) - \epsilon.$$

Then we have

$$\begin{aligned} C^2(\rho) &\leq \left(\sum_i p_i C(|\psi_i\rangle)\right)^2 \\ &= \left(\sum_i \sqrt{p_i} \sqrt{p_i} C(|\psi_i\rangle)\right)^2 \\ &\leq \left(\sum_i p_i\right) \left(\sum_i p_i C^2(|\psi_i\rangle)\right) \\ &\leq \tau(\rho) + \epsilon, \end{aligned}$$

which establishes the inequality $C^2(\rho) \leq \tau(\rho)$ since $\epsilon > 0$ is arbitrary.

Let $\rho_{i,A} = \text{Tr}_B(|\psi_i\rangle\langle\psi_i|)$. One has

$$\begin{aligned} \tau(\rho) &\leq \sum_i p_i C^2(|\psi_i\rangle) \\ &= \sum_i p_i [2(1 - \text{Tr}(\rho_{i,A}^2))] \\ &= 2(1 - \sum_i p_i \text{Tr}(\rho_{i,A}^2)) \\ &\leq 2(1 - \text{Tr}(\rho_A^2)) \end{aligned}$$

due to the convex property of $\text{Tr}(\rho_A^2)$ [9]. □

3. PHC MEASURE: VIEWING CONCURRENCE FROM ANOTHER PERSPECTIVE

In this section, we will establish another entanglement measure—PHC measure which is based on the PHC criterion [25, 26] and show that it coincides with concurrence. Therefore, it provides us an alternative perspective of understanding the concurrence.

It is known that entanglement measures may be induced from some entanglement criteria. For example, negativity and convex-roof extended negativity are two kinds of entanglement measures induced from the elegant PPT criterion [37, 38]. In [26, 25], a necessary and sufficient condition of separability for pure states is proposed. We review some notations firstly.

Definition 2 ([25, 26]). Let $\rho = |\psi\rangle\langle\psi|$ be a pure state acting on $H_A \otimes H_B$ with $\dim H_A \otimes H_B \leq +\infty$ and

$$|\psi\rangle = \sum_k \lambda_k |k\rangle |k'\rangle$$

be the Schmidt decomposition of $|\psi\rangle$. Then

$$\rho = \sum_{k,l} \lambda_k \lambda_l |k\rangle |k'\rangle \langle l| \langle l'|$$

and the partial Hermitian conjugate of ρ is defined by

$$(18) \quad \rho^{\text{PHC}} = \sum_{k,l} \lambda_k \lambda_l |k\rangle |l'\rangle \langle l| \langle k'|.$$

It is showed in [25, 26] that a pure state $\rho = |\psi\rangle\langle\psi|$, $|\psi\rangle \in H_A \otimes H_B$, is separable if and only if $\rho^{\text{PHC}} = \rho$. Consequently, $\rho^{\text{PHC}} \neq \rho$ implies that ρ is entangled, and also that $\|\rho_\psi - \rho_\psi^{\text{PHC}}\|_2 > 0$ (here, $\|\cdot\|_2$ denotes the Hilbert-Schmidt norm, i.e., $\|A\|_2 = [\text{Tr}(A^\dagger A)]^{\frac{1}{2}}$). In what follows, we will show that the PHC criterion does provide us with an entanglement measure.

Definition 3 Let $|\psi\rangle \in H_A \otimes H_B$ with $\dim H_A \otimes H_B \leq +\infty$ be a pure state. The PHC measure of a pure state $|\psi\rangle$ is defined by

$$(19) \quad E_{\text{PHC}}(|\psi\rangle) := \|\rho_\psi - \rho_\psi^{\text{PHC}}\|_2.$$

For mixed state ρ ,

$$(20) \quad E_{\text{PHC}}(\rho) := \inf_{\{p_i, |\psi_i\rangle\}} \left\{ \sum_i p_i E_{\text{PHC}}(|\psi_i\rangle) \right\},$$

where the infimum is taken over all possible ensembles of ρ .

The main result of this section is the following.

Theorem 2 The PHC entanglement measure coincides with the concurrence, i.e., $E_{\text{PHC}}(\rho) = C(\rho)$ for any state ρ acting on $H_A \otimes H_B$ with $\dim H_A \otimes H_B \leq +\infty$.

Proof By the generalized convex roof construction, we only need to show

$$(21) \quad E_{\text{PHC}}(|\psi\rangle) = C(|\psi\rangle)$$

for all pure states $|\psi\rangle$.

Let $|\psi\rangle = \sum_k \lambda_k |k\rangle |k'\rangle$ be the Schmidt decomposition of $|\psi\rangle$. Then $\rho^{\text{PHC}} = \sum_{k,l} \lambda_k \lambda_l |k\rangle |l'\rangle \langle l| \langle k'|$.

Therefore

$$\begin{aligned} & \rho_\psi - \rho_\psi^{\text{PHC}} \\ &= \sum_{k,l} \lambda_k \lambda_l |k\rangle |k'\rangle \langle l| \langle l'| - \sum_{i,j} \lambda_i \lambda_j |i\rangle |j'\rangle \langle j| \langle i'| \\ &= \sum_{k,l} \lambda_k \lambda_l |k\rangle \langle l| \otimes (|k'\rangle \langle l'| - |l'\rangle \langle k'|). \end{aligned}$$

Now,

$$\begin{aligned}
& (\rho_\psi - \rho_\psi^{\text{PHC}})(\rho_\psi - \rho_\psi^{\text{PHC}})^\dagger \\
&= \sum_{k,l,i} \lambda_k \lambda_l^2 \lambda_i |k\rangle\langle i| \otimes (|k'\rangle\langle i'| + \langle k'|i'\rangle |l'\rangle\langle l'| \\
&\quad - \langle l'|i'\rangle |k'\rangle\langle l'| - \langle k'|i'\rangle |i'\rangle\langle i'|)
\end{aligned}$$

implies that

$$\text{Tr}((\rho_\psi - \rho_\psi^{\text{PHC}})(\rho_\psi - \rho_\psi^{\text{PHC}})^\dagger) = 2 \sum_{k \neq l} \lambda_k^2 \lambda_l^2.$$

Therefore

$$E_{\text{PHC}}(|\psi\rangle) = \|\rho_\psi - \rho_\psi^{\text{PHC}}\|_2 = \sqrt{2 \sum_{k \neq l} \lambda_k^2 \lambda_l^2} = C(|\psi\rangle)$$

by Proposition 1, completing the proof. \square

Thus, the PHC measure can also be regarded as a “well-defined” entanglement measure. Although the PHC measure is the same to the concurrence, it shines some new light on the nature of the concurrence.

4. CONCLUSION

Summarizing, the concepts of the concurrence and the tangle for infinite-dimensional bipartite quantum systems are introduced. These two functions are continuous under the trace norm topology. This enables us to prove that the concurrence as well as the tangle are still well-defined monotonic entanglement measures. The relationship between them are discussed and an upper bound is proposed: $C(\rho) \leq \sqrt{\tau(\rho)} \leq \sqrt{2[1 - \text{Tr}(\rho_A^2)]}$, where the equalities hold whenever ρ is a pure state. Based on the partial Hermitian conjugate criterion, the PHC measure is introduced. Moreover, this measure coincides with the concurrence and thus a well defined entanglement measure, which answers a question suggested in [25, 26].

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(Y. Guo) DEPARTMENT OF MATHEMATICS, SHANXI DATONG UNIVERSITY, DATONG 037009, CHINA;, INSTITUTE OF OPTOELECTROINCS ENGINEERING, DEPARTMENT OF PHYSICS AND OPTOELECTROINCS, TAIYUAN UNIVERSITY OF TECHNOLOGY, TAIYUAN 030024, CHINA

E-mail address: guoyu3@yahoo.com.cn

(J. Hou) DEPARTMENT OF MATHEMATICS, TAIYUAN UNIVERSITY OF TECHNOLOGY, TAIYUAN 030024, P. R. CHINA

E-mail address: jinchuanhou@yahoo.com.cn, houjinchuan@tyut.edu.cn

(Y. Wang) INSTITUTE OF OPTOELECTROINCS ENGINEERING, DEPARTMENT OF PHYSICS AND OPTOELECTROINCS, TAIYUAN UNIVERSITY OF TECHNOLOGY, TAIYUAN 030024, CHINA

E-mail address: 2006wang.yinzhu@163.com